Limit Evaluation Techniques – Limits at Infinity

Asymptotes and End Behavior

I think in PreCalculus you talked about horizontal asymptotes and 'end behavior'. You learned one or two rules that worked with the rational functions you studied (and everyone was happy?) In calculus – or any study of functions – we want to extend that work because how a function behaves at large positive (or negative) values of x is interesting. Think of it this way – if the function I'm studying behaves **very much** like a simple rational function (for example) for large values of x, then I can sometimes just use the rational function. In other words, my life got simpler.

True limit thinking – part 2

Two analogous points (to the limits of infinity discussion):

  * Infinity is not a point – so I need a way of figuring out the end behavior without substituting.
  * We know we can pick an arbitrarily large x-value. We want to **know** if the corresponding function value is getting arbitrarily close to a specific real number – no matter how large I pick my x.

As a result we need several techniques we haven't discussed yet.

The basic idea

Consider $\lim\limits_{x\to\infty}\dfrac{x^2-4x+5}{3x^2+18x+100000}$. You learned a rule (really rule of thumb) in PreCalculus that handles this situation, but we are going to learn the 'calculus' way of handling limits to infinity … because we are then going to **extend** that idea to handle other situations.

If we try to substitute, we get really big numbers (infinity if it were a number) on the top and bottom. **No useful information!** So we want to algebraically manipulate this rational function to get a number (as opposed to an x) somewhere. Look at the first term in the numerator. We see that we can cancel out the x^2 then we are left with the number 1.

So our first idea is to divide it by x^2. But we remember our algebra, and realize we must then divide **every** term, **top and bottom**, by the x^2 also.

So …

$$\lim_{x\to\infty}\frac{x^2-4x+5}{3x^2+18x+100000}=\lim_{x\to\infty}\frac{x^2-4x+5}{3x^2+18x+100000}\left(\frac{1/x^2}{1/x^2}\right)=$$

$$\lim_{x\to\infty}\frac{x^2/x^2-4x/x^2+5/x^2}{3x^2/x^2+18x/x^2+100000/x^2}=$$

$$\lim_{x\to\infty}\frac{1-4/x+5/x^2}{3+18/x+100000/x^2}$$

Now notice that we can take the limits of each individual piece. There are a lot of 0s there, but we are left with $\lim\limits_{x\to\infty}\dfrac{1}{3}=\dfrac{1}{3}$. Voilà! Note that this basic idea applies first just to rational functions.

The neat part

This idea works whenever you divide everything by the power of x represented by the degree of the polynomial in the numerator **or** the degree of the polynomial in the denominator. You also need to know the basic limits $\lim_{x\to\infty}\left(\frac{1}{x}\right)$ *and* $\lim_{x\to\infty} x$.

The extension – to radicals

Now consider $\lim_{x\to\infty}\frac{x-4}{\sqrt{3x^2+18x+100000}}$. I want to handle this the same way, but the algebra gets a little more complicated. We will divide by x (degree of top polynomial) and execute some care with the radical.

$$\lim_{x\to\infty}\frac{x-4}{\sqrt{3x^2+18x+100000}} = \lim_{x\to\infty}\frac{x/x-4/x}{\sqrt{3x^2+18x+100000}\,/\,x} =$$

$$\lim_{x\to\infty}\frac{1-4/x}{\sqrt{3x^2+18x+100000}\,/\sqrt{x^2}} = \lim_{x\to\infty}\frac{1-4/x}{\sqrt{3x^2/x^2+18x/x^2+100000/x^2}} = \lim_{x\to\infty}\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

So – to bring the 'divide' by x into the radical sign we must convert it to 'divide' by x^2. This represents a risk, since now $\sqrt{x^2}$ **is always positive**.

The nuance – with radicals

Let's say our limit is now as x approaches **NEGATIVE** infinity: $\lim_{x\to-\infty}\frac{x-4}{\sqrt{3x^2+18x+100000}}$. How do things change? Well, everything in the above string of equations is the same ***until*** I convert the x into $\sqrt{x^2}$**.** When I do that, I've changed the **negative** x into the **positive** $\sqrt{x^2}$**.** So I have to supply the negative sign manually. When I do that, everything is fine. Note that sometimes the answer will change sign; sometimes it won't. I must do the work.

Practice

There is real thinking involved in these problems, even though there are only 3 possible answers (does not exist, +∞, -∞). Or something else (just to be twisted).

1. $\lim_{x\to-\infty}(10-3x)$
2. $\lim_{x\to-\infty}\sqrt{3-x}$
3. $\lim_{x\to-\infty}\frac{5}{4x+16}$
4. $\lim_{x\to-\infty}\frac{7x^2+11x}{2x^2+9x}$
5. $\lim_{x\to+\infty}(\sqrt{x^2+4}-x)$
6. $\lim_{x\to+\infty}\frac{\sqrt{3x^2+2}}{x-7}$